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Convergence of Modified Lagrange Interpolation to L_p-Functions Based on the Zeros of Orthonormal Polynomials with Freud Weights

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We consider the "Freud weight" $W_Q^2(x) = \exp(-Q(x))$. let $1 < p < \infty$, and let $L_n^*(f)$ be a modified Lagrange interpolation polynomial to a measurable function $f \in \{f; \text{ess sup}_{x \in \mathbb{R}} |f(x)|\, W_Q(x)(1+|x|)^{\alpha} < \infty\}$, $\alpha > 0$. Then we have $\lim_{n \to \infty} \int_{-\infty}^{\infty} [|f(x) - L_n^*(f; x)|\, W_Q(x)(1+|x|)^{-\Delta}]^p\, dx = 0$, where Δ is a constant depending on p and α.

1. INTRODUCTION

Let $Q\colon \mathbb{R} \to \mathbb{R}$ be even and continuous on $\mathbb{R}$, Q'' be continuous on $(0, \infty)$, and let $Q' > 0$ on $\mathbb{R}$. Furthermore, we assume that for some A, $B > 1$,

$$A \leqq \{(d/dx)(xQ'(x))\}/Q'(x) \leqq B, \qquad x \in (0, \infty).$$

Then we call this a Freud weight

$$W_Q^2(x) = \exp\{-Q(x)\}. \tag{1.1}$$

The weight $W_\beta^2(x) = \exp\{-|x|^\beta\}$, $\beta > 1$, is a typical example. For $u > 0$, the Mhaskar–Rahmanov–Saff number a_u is defined as the positive root of the equation

$$u = (2/\pi) \int_0^1 a_u t Q'(a_u t)(1-t^2)^{-(1/2)}\, dt, \qquad u > 0. \tag{1.2}$$

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By [6, (5.5)] we have

$$a_n/n \sim 1/Q'(a_n) \to 0 \qquad \text{as} \quad n \to \infty. \tag{1.3}$$

For the weight $W_Q^2(x) = W_\beta^2(x) = \exp\{-|x|^\beta\}$, $\beta > 0$, Mhaskar and Saff [11] show that a_n takes the simple form $a_n = \mu_\beta n^{1/\beta}$, $n = 1, 2, 3, ...$, where μ_β is a constant depending only on β.

Let Π_n denote the class of real polynomials of degree at most n. We define orthonormal polynomials $\{p_n(x)\} = \{p_n(W_Q^2; x)\}$, $p_n \in \Pi_n$, with respect to W_Q^2, that is,

$$\int_{-\infty}^{\infty} p_m(x)\, p_n(x)\, W_Q^2(x)\, dx = \delta_{nm} = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

We denote the zeros of $p_n(x)$ by x_{kn}, $k = 1, 2, ..., n$, where

$$x_{nn} < x_{n-1,n} < \cdots < x_{1n}.$$

Let $L_n(f) \in \Pi_{n-1}$ be the Lagrange interpolation polynomial to f at the zeros $\{x_{kn}\}$ of $p_n(W_Q^2; x)$, which is defined to be a unique polynomial such that

$$L_n(f; x) = \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x),$$

where the fundamental polynomials l_{kn} are defined by

$$l_{kn}(x) = p_n(x)/\{(x - x_{kn})\, p_n'(x_{kn})\}, \qquad k = 1, 2, ..., n.$$

Nevai [13] showed the following.

THEOREM A. *Let f be a continuous function on $\mathbb{R}$. Assume that f satisfies*

$$\lim_{|x| \to \infty} f(x)(1 + |x|) \exp(-x^2/2) = 0.$$

Then

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} [\,|f(x) - L_n(f; x)| \exp(-x^2/2)]^p\, dx = 0$$

holds for every $p > 1$.

Knopfmacher and Lubinsky [4] obtained an extension of Theorem A, which relates to a general Freud weight, and they also investigated the approximation of some function with finitely many singularities by certain modified Lagrange interpolation polynomials. Furthermore, Lubinsky and Matjila [7] provided the following nice result.

THEOREM B. *Let the weight* $W_Q^2(x)$ *be defined by* (1.1), *and let* $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$, *and* $\hat{\alpha} = \min(\alpha, 1)$. *Then for*

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} [|f(x) - L_n(f; x)| \, W_Q(x)(1 + |x|)^{-\Delta}]^p \, dx = 0,$$

to hold for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\lim_{|x| \to \infty} |f(x)| \, W_Q(x)(1 + |x|)^{\alpha} = 0, \tag{1.4}$$

if $p \leqq 4$, *it is necessary and sufficient that*

$$\Delta > -\hat{\alpha} + (1/p);$$

and if $p > 4$ *and* $\alpha \neq 1$, *it is necessary and sufficient that*

$$a_n^{1/p - (\hat{\alpha} + \Delta)} > n^{(1/6)(1 - 4/p)} = 0(1), \qquad n \to \infty;$$

and if $p > 4$ *and* $\alpha = 1$, *it is necessary and sufficient that*

$$a_n^{1/p - (\hat{\alpha} + \Delta)} > n^{(1/6)(1 - 4/p)} = 0(1/\log n), \qquad n \to \infty.$$

Let $1 < p < \infty$. Our purpose in this paper is to approximate certain functions which are not always continuous by our modified Lagrange interpolation polynomials. To obtain our theorem we shall apply the Lubinsky and Matjila result for a continuous function.

We need some classes of functions on $\mathbb{R}$. Let

$$S = \Big\{ s;\, s(t) = \sum_{i=1}^{m} s_i \chi(I_i; t),\ \bigcup_{i=1}^{m} I_i = [c, d),\ I_i = [c_i, d_i),$$

$$c_{i+1} = d_i,\ s_i \in \mathbb{R}\ (i = 1, 2, ..., m),\ 1 \leqq m < \infty,\ -\infty < c < d < \infty \Big\}, \tag{1.5}$$

where $\chi(I_i; t)$ is the characteristic function on I_i. Then for each $\alpha > 0$ we define the class $E(\alpha, W_Q)$ by

$$E(\alpha, W_Q) = \{ f;\ \text{for each } \varepsilon > 0 \text{ there exists } s \in S \text{ such that}$$

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x) - s(x)| \, W_Q(x)(1 + |x|)^{\alpha} < \varepsilon \}. \tag{1.6}$$

If $f \in C(\mathbb{R})$ satisfies (1.4), then $f \in E(\alpha, W_Q)$, and if

$$f(x) = \sum_{i=1}^{\infty} f_i \chi(I_i; x), \qquad I_i \cap I_j = \varnothing \quad (i \neq j),\ f_i \in \mathbb{R} \quad (i = 1, 2, 3, ...),$$

and for each $\varepsilon > 0$ there exists $a(\varepsilon) > 0$ such that

$$|f(x)|\, W_Q(x)(1 + |x|)^{\alpha} < \varepsilon, \qquad |x| \geqq a(\varepsilon),$$

then $f \in E(\alpha, W_Q)$.

Let f be integrable on any compact interval $[a, b]$. Then we define a modified Lagrange interpolation polynomial $L_n^*(f; x)$ by

$$L_n^*(f; x) = \sum_{k=1}^{n} f_n(x_{kn})\, l_{kn}(x),$$

where $\varepsilon_n = \delta a_n/n$ for a certain constant $\delta > 0$ small enough and

$$f_n(x_{kn}) = (1/\varepsilon_n) \int_{\varepsilon_{kn}^-}^{\varepsilon_{kn}^+} f(x_{kn} + t)\, dt,$$
$$\varepsilon_{kn}^+ = \begin{cases} 0 & \text{for} \quad x_{kn} \geqq 0, \\ \varepsilon_n & \text{for} \quad x_{kn} < 0, \end{cases} \qquad \varepsilon_{kn}^- = \begin{cases} -\varepsilon_n & \text{for} \quad x_{kn} \geqq 0, \\ 0 & \text{for} \quad x_{kn} < 0. \end{cases} \tag{1.7}$$

The function

$$f_n(x) = (1/\varepsilon_n) \int_{\varepsilon_n^-}^{\varepsilon_n^+} f(x + t)\, dt,$$
$$\varepsilon_n^+ = \begin{cases} 0 & \text{for} \quad x \geqq 0, \\ \varepsilon_n & \text{for} \quad x < 0, \end{cases} \qquad \varepsilon_n^- = \begin{cases} -\varepsilon_n & \text{for} \quad x \geqq 0, \\ 0 & \text{for} \quad x < 0, \end{cases}$$

gives a mean value of f in the neighborhood $(x - \varepsilon_n, x)$ or $(x, x + \varepsilon_n)$. If f is continuous at x, then by (1.3) we have $f_n(x) \to f(x)$ as $n \to \infty$. Here our modified Lagrange interpolation polynomial $L_n^*(f; x)$ satisfies $L_n^*(f; x_{kn}) = L_n(f_n; x_{kn}) = f_n(x_{kn})$.

Now, our theorem is the following.

THEOREM. *Let $1 < p < \infty$. Then for every function $f \in E(\alpha, W_Q)$ and each number Δ satisfying the condition in Theorem B, we have*

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} [|f(x) - L_n^*(f; x)|\, W_Q(x)(1 + |x|)^{-\Delta}]^p\, dx = 0. \tag{1.8}$$

2. NOTATIONS AND PRELIMINARIES

Throughout this paper $c_1, c_2, \ldots$ will denote positive constants independent of n and x, and the letter c denotes a constant which may differ at each different occurrence, even in the same chain of inequalities. Let $c(a, b, \ldots)$ mean a constant depending on $a, b, \ldots$. We need some lemmas.

For a constant $M > 0$, we define a subset S_M of S in (1.5) by

$$S_M = \{s \colon s \in S,\ |s(x)|\ W_Q(x)(1+|x|)^\alpha \leqq M\ (x \in \mathbb{R})\}. \tag{2.1}$$

Let $f \in E(\alpha, W_Q)$ be defined by (1.6). Then we see

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)|\ W_Q(x)(1+|x|)^\alpha = M_f < \infty. \tag{2.2}$$

Let $0 < \varepsilon < 1$. Then we find a step function $s \in S_M$, where $M = M_f + 1$, satisfying

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x) - s(x)|\ W_Q(x)(1+|x|)^\alpha < \varepsilon. \tag{2.3}$$

We need to construct a continuous function $f_{s\varepsilon} \in C(\mathbb{R})$. Put

$$a_u = \max\{|c_1 - 1|, |d_m + 1|\},$$

where $[c_1, d_m]$ is the compact support of $s \in S_M$, and a_u is Mhaskar–Rahmanov–Saff number in (1.2). Let

$$0 < \kappa < (1/4) \min[a_u/u, |c_i - d_i|\ (i = 1, 2, 3, \ldots, m)].$$

Now, we define the continuous function $f_{s\varepsilon} \in C(\mathbb{R})$ by

$$f_{s\varepsilon}(x) = \begin{cases} \text{linear} & \begin{cases} \text{on } [d_i, d_i+\kappa] & \text{for } |s(c_i)| < |s(d_i+\kappa)|, \text{ or} \\ \text{on } [d_i-\kappa, d_i] & \text{for } |s(d_i+\kappa)| < |s(c_i)|, \\ & \quad i = 1, 2, 3, \ldots, m, \qquad \text{and} \\ \text{on } [c_1-\kappa, c_1],\ [d_m, d_m+\kappa], & \end{cases} \\ s(x) & \text{for otherwise,} \end{cases} \tag{2.4}$$

and we set

$$E_s = \{x; f_{s\varepsilon}(x) \neq s(x), x \in \mathbb{R}\}, \qquad E_s^* = \{x; x = t + u\kappa, |u| \leqq 1, t \in E_s\}. \tag{2.5}$$

Here we suppose

$$\text{Meas. } E_s^* = 3(m+1)\,\kappa < \varepsilon < 1. \tag{2.6}$$

For each $f_{s\varepsilon}$ we consider the function

$$\Psi_n(x) = \Psi_n(f_{s\varepsilon}; x) = (1/\varepsilon_n) \int_{\varepsilon_{kn}^-}^{\varepsilon_{kn}^+} \{f_{s\varepsilon}(x) - f_{s\varepsilon}(x+t)\}\, dt, \qquad (2.7)$$

where $\varepsilon_n = \delta a_n/n$, and ε_{kn}^+, ε_{kn}^- are defined in (1.7). By the simple observation we obtain the following lemma with respect to the function Ψ_n.

LEMMA 2.1. *Let the conditions* (2.1)–(2.6) *be satisfied. Then*

$$\text{(i)} \quad \Psi_n(x) = 0, \qquad x \notin E_s^*, \qquad (2.8)$$

$$\text{(ii)} \quad |\Psi_n(x)| \leqq (M_f M_{Q\alpha}/\kappa)\, \varepsilon_n \{W_Q(x)(1+|x|)^\alpha\}^{-1}, \qquad x \in \mathbb{R}, \qquad (2.9)$$

where $M = M_f + 1$ *for* M_f *in* (2.2), *and* $M_{Q\alpha} = \sup_{x \in \mathbb{R}} W_Q(x)(1+|x|)^\alpha$.

Proof. Let $[c_1, d_m]$ be the compact support of $s \in S_M$, where $M = M_f + 1$. By the definition of $f_{s\varepsilon}$

$$\Psi_n(x) = 0, \qquad x \notin E_s^*,$$

that is, we have (2.8).

The definition of $f_{s\varepsilon}$ means $|f_{s\varepsilon}(x+t) - f_{s\varepsilon}(x)| \leqq (2M_f/\kappa)\,|t|$ for $x \in \mathbb{R}$. Therefore, we have

$$\begin{aligned} &|\Psi_n(x)|\, W_Q(x)(1+|x|)^\alpha \\ &\quad \leqq (1/\varepsilon_n) \int_{\varepsilon_{kn}^-}^{\varepsilon_{kn}^+} |f_{s\varepsilon}(x) - f_{s\varepsilon}(x+t)|\, dt\, W_Q(x)(1+|x|)^\alpha \\ &\quad \leqq (2M_f/\kappa)(1/\varepsilon_n) \int_0^{\varepsilon_n} t\, dt\, W_Q(x)(1+|x|)^\alpha \\ &\quad \leqq (M_f M_{Q\alpha}/\kappa)\, \varepsilon_n. \end{aligned}$$

Consequently, we have (2.9). ▮

For $f \in E(\alpha, W_Q)$ and $\varepsilon > 0$ we consider the function $f_{s\varepsilon} \in C(\mathbb{R})$ in (2.4). Let Ψ_n be defined by (2.7), and let us take $0 < \sigma < 1/2$ and κ small enough satisfying (2.6). The inequality (2.9) means that for $n \geqq n(f_{s\varepsilon}, Q, \alpha, \delta)$ large enough

$$|\Psi_n(x)|\, W_Q(x) \leqq \varepsilon(1+|x|)^{-\alpha}. \qquad (2.10)$$

The following two lemmas are obtained by Lubinsky and Matjila (see (2.8) and (2.10)).

LEMMA 2.2 [7, Lemma 3.4]. *Let Δ be defined in Theorem* B. *For each* $1<p<\infty$ *we have*

$$\limsup_{n\to\infty}\int_{-\sigma a_n}^{\sigma a_n}|L_n(\Psi_n;x)\,W_Q(x)(1+|x|)^{-\Delta}|^p\,dx\leqq c\varepsilon,$$

where c is independent of ε, *n, and* $\{\Psi_n(x)\}$.

LEMMA 2.3 [7, Lemma 3.2]. *Let Δ be defined in Theorem* B. *For each* $1<p<\infty$ *we have*

$$\limsup_{n\to\infty}\int_{|x|\geqq\sigma a_n}|L_n(\Psi_n;x)\,W_Q(x)(1+|x|)^{-\Delta}|^p\,dx\leqq c\varepsilon,$$

where c is independent of ε, *n, and* $\{\Psi_n(x)\}$.

3. PROOF OF THE THEOREM

Let $f\in E(\alpha, W_Q)$, and let $0<\varepsilon<1$. Then there exists $s\in S$ such that

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}}|f(x)-s(x)|\,W_Q(x)(1+|x|)^{\alpha}\leqq\varepsilon. \tag{3.1}$$

Thus we see that there exists $a(\varepsilon)>0$ such that

$$\operatorname*{ess\,sup}_{|x|\geqq a(\varepsilon)}|f(x)|\,W_Q(x)(1+|x|)^{\alpha}\leqq\varepsilon.$$

Especially for $\varepsilon=1$ there exists $M_f>0$ such that

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}}|f(x)|\,W_Q(x)(1+|x|)^{\alpha}\leqq M_f<\infty. \tag{3.2}$$

If we put $M=M_f+1$, then by (3.1) and (3.2) we obtain $s\in S_M$. Applying Lemmas 2.2 and 2.3, we see

$$\limsup_{n\to\infty}\int_{-\infty}^{\infty}|L_n(\Psi_n;x)\,W_Q(x)(1+|x|)^{-\Delta}|^p\,dx\leqq c\varepsilon. \tag{3.3}$$

By (3.1)

$$\int_{-\infty}^{\infty}\{|f(x)-s(x)|\,W_Q(x)(1+|x|)^{-\Delta}\}^p\,dx$$

$$\leqq\varepsilon^p\int_{-\infty}^{\infty}(1+|x|)^{-(\Delta+\alpha)p}\,dx\leqq c\varepsilon^p. \tag{3.4}$$

We can show that for each $x \in \mathbb{R}$

$$|s(x) - f_{s\varepsilon}(x)| \; W_Q(x) \leqq 2(M_f + 1) \; M_{Q\alpha}(1 + |x|)^{-\alpha},$$

where M_f and $M_{Q\alpha}$ are the constants in Lemma 2.1 (by the definition of $f_{s\varepsilon}$ we remark $|f_{s\varepsilon}(x)| \leqq |s(x)|$, $x \in \mathbb{R}$). Hence we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \{|s(x) - f_{s\varepsilon}(x)| \; W_Q(x)(1 + |x|)^{-\Delta}\}^p \, dx \\
&\quad \leqq 2(M_f + 1) \; M_{Q\alpha} \int_{E_s^*} (1 + |x|)^{-(\Delta + \alpha) p} \, dx \\
&\quad \leqq c(M_f + 1) \; M_{Q\alpha} \varepsilon. \qquad (3.5)
\end{aligned}$$

By (3.4) and (3.5) we see that

$$\int_{-\infty}^{\infty} |\{f(x) - f_{s\varepsilon}(w)\} \; W_Q(x)(1 + |x|)^{-\Delta}\}|^p \, dx \leqq c\varepsilon. \qquad (3.6)$$

Thus by (3.3), (3.6), and Theorem B we have

$$\begin{aligned}
&\left[\int_{-\infty}^{\infty} |\{f(x) - L_n^*(f; x)\} \; W_Q(x)(1 + |x|)^{-\Delta}|^p \, dx\right]^{1/p} \\
&\quad \leqq \left[\int_{-\infty}^{\infty} |\{f(x) - f_{s\varepsilon}(x)\} \; W_Q(x)(1 + |x|)^{-\Delta}|^p \, dx\right]^{1/p} \\
&\qquad + \left[\int_{-\infty}^{\infty} |\{f_{s\varepsilon}(x) - L_n(f_{s\varepsilon}; x)\} \; W_Q(x)(1 + |x|)^{-\Delta}|^p \, dx\right]^{1/p} \\
&\qquad + \left[\int_{-\infty}^{\infty} \left|L_n\left(f_{s\varepsilon} - (1/\varepsilon_n) \int_{\varepsilon_{kn}^-}^{\varepsilon_{kn}^+} f_{s\varepsilon}(\cdot + t) \, dt; x\right) W_Q(x)(1 + |x|)^{-\Delta}\right|^p dx\right]^{1/p} \\
&\quad \leqq c\varepsilon,
\end{aligned}$$

for n large enough. Since $\varepsilon > 0$ is arbitrary, we have (1.8).

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